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Additional symmetries of generalized integrable hierarchies

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Abstract. The non-isospectral symmetries of a general class of integrable hierarchies are found by generalizing the Galilean and scaling symmetries of the Korteweg–de Vries equation and its hierarchy. The symmetries arise in a very natural way from the semi-direct product structure of the Virasoro algebra and the affine Kac–Moody algebra underlying the construction of the hierarchy. In particular, the generators of the symmetries are shown to satisfy a subalgebra of the Virasoro algebra. When a tau-function formalism is available, the infinitesimal symmetries act directly on the tau-functions as moments of Virasoro currents. Some comments are made regarding the role of the non-isospectral symmetries and the form of the string equations in matrix-model formulations of quantum gravity in two dimensions and related systems.

1. Introduction

Since the discovery of the soliton solution of the Korteweg–de Vries (KdV) equation

$$\frac{\partial u}{\partial t} = \frac{1}{4}\frac{\partial^3 u}{\partial x^3} + \frac{3}{2}u\frac{\partial u}{\partial x} \tag{1.1}$$

much effort has been devoted to elucidating the nature of the integrability of soliton hierarchies (see [1] for a nice discussion on the history of the theory of soliton equations, and [2] for an introduction to integrable models). One of the more important and surprising developments has been the recognition of the deep connection existing between integrable hierarchies of nonlinear differential equations and infinite-dimensional Lie algebras. This connection manifested itself in two apparently unconnected approaches.

In the 'tau-function' approach, pioneered by the Japanese school [3,4], the equations are cast in a particular bilinear or Hirota form by the use of a special set of variables—the tau-functions. For instance, the original variable and the tau-function of the KdV equation are related—in standard convention—by the well known formula

$$u = 2\frac{\partial^2}{\partial x^2}\ln\tau. \tag{1.2}$$

It was clear, in the original work of [4], that the affine Kac–Moody algebras play a central role in this approach but it was made even clearer by Kac and Wakimoto [5]. In this last

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work, the authors constructed integrable hierarchies of equations in Hirota form associated with vertex-operator representations of affine Kac–Moody algebras.

The other approach is inspired by the seminal work of Drinfel'd and Sokolov [6]. In their construction, the central objects are gauge fields in the loop-algebra of a finite Lie algebra and the equations are the conditions of zero-curvature on these gauge fields. In the original work of [6], the authors make use of the 'principal' gradation of the loop-algebra in an essential way; in particular, the construction involves the 'principal Heisenberg subalgebra'. On the other hand, it is well known that the affine Kac–Moody algebras have other inequivalent Heisenberg subalgebras [7, 8], an observation that was exploited in [9] (see also [10]) to construct more general integrable hierarchies. In fact, it was shown in [11] that it is not necessary to be restricted to the loop-algebra and the constructions of the hierarchies are completely representation independent; in particular, they are independent of the centre. Moreover, when the affine Kac–Moody algebra has a vertex-operator representation the zero-curvature hierarchies can be written in terms of tau-functions [11] and hence a bridge can be established between the two approaches.

The purpose of this paper is to complete the description of the zero-curvature integrable hierarchies of [9] by discussing their 'additional' or 'non-isospectral' symmetries†. In this context, a 'symmetry' is some transformation that relates one solution of an equation with another solution of the same equation. Recall that, for example, the KdV equation has an infinite set of commuting symmetries, infinitesimally generated by the infinite set of conserved Hamiltonians. Hence, because the existence of an infinite set of conserved Hamiltonians is a necessary condition for integrability, any integrable nonlinear differential equation has an infinite set of commuting symmetries; these are actually the 'isospectral' symmetries. The corresponding infinite set of infinitesimal generators are nothing but the commuting flows defining the 'integrable hierarchy' associated with the original nonlinear differential equation. In the case of the KdV equation, the general form of these flows is

$$\frac{\partial u}{\partial t_{2k+1}} = P_k\left(u, \partial_x u, \partial_x^2 u, \ldots\right) \tag{1.3}$$

where $t_1 \equiv x$, $t_3 \equiv t$ and P_k is a polynomial function of u and its x-derivatives; the original KdV equation corresponding to $k = 1$. For each integer $k > 1$, (1.3) generates a symmetry of the KdV equation in the sense that if u is a solution then so is $u + \epsilon\partial u/\partial t_{2k+1}$ $\epsilon \ll 1$.

However, the group of symmetries of the KdV equation is known to be much bigger still, since there exist the Galilean and scaling transformations

$$\begin{aligned} u(x,t) &\mapsto \widetilde{u}(x,t) = u(x+vt,t) + \tfrac{2}{3}v \\ u(x,t) &\mapsto \widetilde{u}(x,t) = \mathrm{e}^{2r}u(\mathrm{e}^{r}x, \mathrm{e}^{3r}t) \end{aligned} \tag{1.4}$$

respectively, for arbitrary values of v and r. When v and r are infinitesimal, one can write them as

$$\widetilde{u}(x,t) - u(x,t) \approx \frac{2v}{3}\left(\frac{3}{2}t\frac{\partial u}{\partial x} + 1\right) \tag{1.5}$$

$$\widetilde{u}(x,t) - u(x,t) \approx r\left(x\frac{\partial u}{\partial x} + 3t\frac{\partial u}{\partial t} + 2u\right). \tag{1.6}$$

† A symmetry is called either 'isospectral' or 'non-isospectral' according to whether it preserves or changes the spectrum of the auxiliary linear problem associated with the nonlinear equation, respectively [12, 13].

Notice that, in contrast to the flows defined by (1.3), the right-hand sides of (1.5) and (1.6) involve x and t explicitly in addition to u and its x-derivatives (hence, the flows cannot be Hamiltonian). (1.5) and (1.6) are the first two generators of the infinite set of 'non-isospectral' or 'additional' symmetries of the KdV equation. These additional symmetries commute with the isospectral ones, i.e. the original flows of the KdV hierarchy, but do not commute among themselves. Instead, their generators satisfy part of the Virasoro algebra; the two transformations in (1.5) and (1.6) corresponding to the L_{-1} and L_0 generators, respectively. The connection between additional symmetries and the Virasoro algebra can be seen most clearly at the level of the tau-functions as discussed in [14] where it has been shown that they are generated by the infinitesimal transformations of the tau-function

$$\tau \mapsto \tilde{\tau} = \tau + \epsilon L_m \tau \tag{1.7}$$

with $m \geqslant -1$ and L_m being the generators of the Virasoro algebra acting in the appropriate Fock space.

The additional (non-isospectral) symmetries of the KdV equation were first found in [15]. Similar symmetries have also been found in the Kadomtsev–Petviashvili (KP) equation and its hierarchy [16] and in the Ablowitz–Kaup–Newell–Segur (AKNS) system [17] leading to a general construction of non-isospectral symmetries of integrable hierarchies by the inverse-scattering method in [18] (see [13] for a more complete list of references). In all these cases, the generators of the additional symmetries satisfy a subalgebra of the Virasoro algebra suggesting that there is a natural action of (part of) the Virasoro algebra, or Diff(S^1), on integrable hierarchies.

More recently, there has been a renewed interest in the integrable hierarchies of partial-differential equations and their additional (non-isospectral) symmetries because of the important role they play in the matrix-model formulation of two-dimensional quantum and topological gravity (see [19] and references therein). In the continuum, or *double-scaling* limit, the partition function of the matrix models is a tau-function of an integrable hierarchy†. Furthermore, this tau-function is constrained by an additional equation known as the 'string equation' which turns out to be the condition that the relevant solution of the hierarchy is actually invariant under the additional symmetries [12]. As has been said previously, the generators of the additional symmetries satisfy a subalgebra of the Virasoro algebra and this is the reason why the string equation appears as a set of Virasoro constraints on the matrix-model partition function [20, 21]. On a slightly different tack, the Virasoro constraints also encode the contact terms of physical operators [22] in some two-dimensional topological field theories. These issues have motivated an investigation into the additional symmetries of the Drinfel'd–Sokolov A_n-KdV hierarchies [12, 23], and those of the KP hierarchy in [23, 24].

In this paper, we shall discuss the additional symmetries of the zero-curvature hierarchies of [9–11] (some preliminary results have been presented in [25]). It turns out that the origin of the additional symmetries and of the Virasoro action on the hierarchy is very natural. It is induced precisely by the semi-direct product of the Virasoro algebra with the affine Kac–Moody algebra in terms of which the hierarchy is defined. Furthermore, both the flows of the zero-curvature hierarchies and the generators of their additional symmetries are constructed in terms of the affine Kac–Moody algebra in a completely representation-independent way. It is also shown that the action of the symmetries on the tau-functions is precisely that of (1.7). In order to make the paper reasonably self-contained, we summarize in section 2

† This also seems to be true, in some cases, before taking the double-scaling limit.

the main results of [9–11]. The expression for the generators of the additional symmetries are found in section 3 and they are shown to satisfy a subalgebra of the Virasoro algebra. Section 4 connects the results to the tau-function formalism, recovering the results of [14]. Finally, in section 5, we propose a generalization of the 'string equation' that selects the solution that is invariant under the additional symmetries. This can be taken as the starting point of an investigation into the possible relation between the generalized hierarchies and two-dimensional physical systems including quantum gravity. Our conventions and some properties of affine Kac–Moody algebras are presented in the appendix.

2. The generalized zero-curvature hierarchies

In this section, we summarize the main results of [9, 11], to which one should refer for further details. Our conventions concerning affine Kac–Moody algebras are summarized in the appendix of [11].

In [9], a generalized integrable hierarchy was associated with each untwisted affine Kac–Moody algebra $\mathfrak{g}$, a particular Heisenberg subalgebra $\mathfrak{s} \subset \mathfrak{g}$ (whose associated gradation is s') and an additional gradation s such that $s \preceq s'$ with respect to a partial ordering (see the appendix).

There is a flow of the hierarchy for each element of $\mathfrak{s}$ of non-negative s'-grade. This is the set $\{b_j,\ j \in E \geqslant 0\}$. The flows are defined in terms of 'Lax operators' of the form

$$\mathcal{L}_j = \frac{\partial}{\partial t_j} - b_j - q(j) \qquad j \in E \geqslant 0 \tag{2.1}$$

where $q(j)$ is a function of the t_js on the intersection

$$Q(j) = \mathfrak{g}_{\geqslant 0}(s) \cap \mathfrak{g}_{<j}(s'). \tag{2.2}$$

Here an expression like $\mathfrak{g}_{<j}(s)$ means the subspace of $\mathfrak{g}$ with s-grade less than j. In order to ensure that the flows are uniquely associated to elements of the set $\{b_j, j \in E \geqslant 0\}$, we will demand that $q(j)$ has no constant terms proportional to b_i with $i < j$. The integrable hierarchy of equations is defined by the zero-curvature conditions

$$[\mathcal{L}_i, \mathcal{L}_j] = 0. \tag{2.3}$$

In general, the above system of equations exhibits a *gauge invariance* of the form

$$\mathcal{L}_j \to U\mathcal{L}_j U^{-1} \tag{2.4}$$

preserving $q(j) \in Q(j)$ where U is a function on the group generated by the finite-dimensional subalgebra given by the intersection

$$P = \mathfrak{g}_0(s) \cap \mathfrak{g}_{<0}(s'). \tag{2.5}$$

The equations of the hierarchy are to be thought of as equations on the equivalence of classes of $Q(j)$ under these gauge transformations. The gauge transformations include the case when U is just a function (related to the exponentiation of the centre of $\mathfrak{g}$). These last transformations can be used to set the component of $q(j)$ in the centre of $\mathfrak{g}$, $q_c(j)$ to any arbitrary value; thus showing that it is not a dynamic degree of freedom but only a purely gauge-dependent quantity so that the hierarchy is completely independent of the centre of $\mathfrak{g}$.

A convenient choice for the gauge slice has been proposed in [11] and we shall use it in the following. With this gauge choice, the integrable hierarchy can be defined in terms of some $\Theta \in U_-(s)$ through the equations

$$\mathcal{L}_j = \frac{\partial}{\partial t_j} + \Theta\left(\frac{\partial}{\partial t_j} - b_j\right)\Theta^{-1} \tag{2.6}$$

and

$$\frac{\partial \Theta}{\partial t_j} = -\mathrm{P}_{<0[s]}(\Theta b_j \Theta^{-1})\Theta \qquad j \in E \geqslant 0. \tag{2.7}$$

Therefore, by comparison with (2.1),

$$q(j) = \mathrm{P}_{\geqslant 0[s]}(\Theta b_j \Theta^{-1}) - b_j \in Q(j). \tag{2.8}$$

Now, the zero-curvature conditions (2.3) are just a consequence of the trivial identities

$$\left[\frac{\partial}{\partial t_i} - b_i, \frac{\partial}{\partial t_j} - b_j\right] = 0 \qquad i, j \in E \geqslant 0. \tag{2.9}$$

Equations (2.3) can be written in Hamiltonian form [10].

3. The additional symmetries

The derivation of the additional symmetries of the hierarchy closely follows the arguments of [9] used to construct the flows of the hierarchy. The idea is to obtain new operators which commute with the Lax operators $\mathcal{L}_i$ but now allow for explicit dependence on the t_js. In particular, using the semi-direct product of the Virasoro algebra with $\mathfrak{g}$ (see the appendix), we shall find an infinite number of operators commuting with the $\mathcal{L}_i$s that depend on the t_js. Our procedure is quite similar to that of [18] (see also [26] and [23]). As before, let us consider an affine Kac–Moody algebra $\mathfrak{g}$, a given Heisenberg subalgebra $\mathfrak{s} \subset \mathfrak{g}$ (whose associated gradation is s') and an additional gradation $s \preceq s'$. In what follows $N_s = \sum_{i=0}^{r} k_i s_i$ where k_i are the Kac labels of $\mathfrak{g}$. We shall need the following two lemmas.

Lemma 3.1. If $[\mathfrak{d}_n^{(s)} + M, \mathcal{L}_j] = 0$ where $M \in \mathfrak{g}$ and

$$n \in \mathbb{Z} \geqslant \begin{cases} -1 & \text{if } s = s_{\mathrm{hom}} \\ 0 & \text{if } s \succ s_{\mathrm{hom}} \end{cases} \tag{3.1}$$

then

$$[\mathfrak{d}_n^{(s)} + \mathrm{P}_{\geqslant 0[s]}(M), \mathcal{L}_j] = -[\mathrm{P}_{<0[s]}(M), \mathcal{L}_j] \in Q(j). \tag{3.2}$$

Proof. The proof of this lemma proceeds by equating the grades of the left- and right-hand sides of (3.2) and taking into account that $[\mathfrak{d}_n^{(s)}, \mathfrak{g}_0(s)] = 0$ and $[\mathfrak{d}_n^{(s)}, \mathfrak{g}_{\geqslant 0}(s)] \subseteq \mathfrak{g}_{\geqslant nN_s+1}(s)$. With this in mind, the s'-grade of the right-hand side is $< j$ and the s-grade of the left-hand side is $\geqslant \min(0, nN_s + 1)$. Since both sides must have the same grade, they actually lie in the intersection $Q(j)$ if the value of n is restricted by (3.1) (recall that $N_s = 1$ only if $s = s_{\mathrm{hom}}$; otherwise, $N_s > 1$). □

Lemma 3.2. For any $n \in \mathbb{Z}$, let us define

$$S_n = \Theta \widetilde{s}_n \Theta^{-1} - \mathfrak{d}_n^{(s)} \in \mathfrak{g} \tag{3.3}$$

where

$$\widetilde{s}_n = \mathfrak{d}_n^{(s')} + \sum_{j \in E > 0} \frac{j}{N_{s'}} t_j b_{j+nN_{s'}} + \frac{\alpha}{N_{s'}} b_{nN_{s'}} \\ + c\left(\left[\frac{1}{2N_{s'}} \sum_{\substack{j+k+nN_{s'}=0 \\ j,k \in E>0}} jkt_j t_k + |n| \alpha t_{|n|N_{s'}} \right] \theta(-n) + \lambda \delta_{n,0} \right) \tag{3.4}$$

where α and λ are arbitrary constants (α is not present if $0 \notin E$) and $\theta(x) = 1$ if $x > 0$, otherwise vanishing (notice that $S_n \in \mathfrak{g}$ because $\mathfrak{d}_n^{(s)} - \mathfrak{d}_n^{(s')} \in \mathfrak{g}$, see (A.4)). Then,

(i) for any $n \in \mathbb{Z}$ and $j \in E \geqslant 0$

$$[\mathfrak{d}_n^{(s)} + S_n, \mathcal{L}_j] = 0 \tag{3.5}$$

(ii) for any $m, n \in \mathbb{Z}$

$$[\widetilde{s}_m, \widetilde{s}_n] = (m - n)\left(\widetilde{s}_{m+n} - c\left[\lambda - \frac{\alpha^2}{2N_{s'}} \right] \delta_{m+n,0} \right) + \frac{\widetilde{c}_v}{12} m(m^2 - 1)\delta_{m+n,0}. \tag{3.6}$$

Proof. The proof of (i) follows from (2.6) and the identity

$$\left[\widetilde{s}_n, \frac{\partial}{\partial t_j} - b_j \right] = 0. \tag{3.7}$$

The proof of (ii) is also straightforward by using (A.2) and

$$[\mathfrak{d}_m^{(s')}, b_j] = -\frac{j}{N_{s'}} b_{j+mN_{s'}}. \tag{3.8}$$

We now define the following set of derivations:

$$\frac{\partial q(j)}{\partial \beta_n} = -[\mathrm{P}_{<0[s]}(S_n), \mathcal{L}_j] \\ = +[\mathfrak{d}_n^{(s)} + \mathrm{P}_{\geqslant 0[s]}(S_n), \mathcal{L}_j] \qquad n \in \mathbb{Z} \geqslant \begin{cases} -1 & \text{if } s = s_{\text{hom}} \\ 0 & \text{if } s \succ s_{\text{hom}} \end{cases} \tag{3.9}$$

where the equality follows from lemma 3.1. These derivations act on Θ as

$$\frac{\partial \Theta}{\partial \beta_n} = \mathrm{P}_{<0[s]}(S_n)\Theta. \tag{3.10}$$

Obviously, as shown in lemma 3.1, $\partial q(j)/\partial \beta_n \in Q(j)$ for the values of n indicated in (3.9); moreover, (3.10) is consistent with our gauge choice $\Theta \in U_-(s)$.

The derivations in (3.9) actually define symmetries of the hierarchy since, as we shall prove in the following proposition, they commute with the flows of the hierarchy.

Proposition 3.3. The flows defined by (2.3) and the derivations defined by (3.9) commute.

Proof. It will be sufficient to show that the flows and the new derivations commute acting on a generic Lax operator. Therefore, let us consider $i, j \in E \geqslant 0$. Then

$$\begin{aligned}\left[\frac{\partial}{\partial t_i}, \frac{\partial}{\partial \beta_n}\right]\mathcal{L}_j &= [\mathrm{P}_{<0[s]}(\Theta b_i \Theta^{-1}), [\mathrm{P}_{<0[s]}(S_n), \mathcal{L}_j]] \\ &\quad + [\mathrm{P}_{<0[s]}([\mathrm{P}_{<0[s]}(S_n), \Theta b_i \Theta^{-1}]), \mathcal{L}_j] \\ &\quad + \left[\mathrm{P}_{<0[s]}\left(\Theta \frac{\partial \widetilde{s}_n}{\partial t_i}\Theta^{-1}\right), \mathcal{L}_j\right] - [\mathrm{P}_{<0[s]}(S_n), [\mathrm{P}_{<0[s]}(\Theta b_i \Theta^{-1}), \mathcal{L}_j]] \\ &\quad - [\mathrm{P}_{<0[s]}([\mathrm{P}_{<0[s]}(\Theta b_i \Theta^{-1}), \Theta \widetilde{s}_n \Theta^{-1}]), \mathcal{L}_j] \\ &= -\left[\mathrm{P}_{<0[s]}\left(\Theta\left[\frac{\partial}{\partial t_i} - b_i, \widetilde{s}_n\right]\Theta^{-1}\right), \mathcal{L}_j\right] \end{aligned} \tag{3.11}$$

which vanishes because of (3.7). □

Therefore, the new derivations (3.9) actually generate an infinite set of additional symmetries of the integrable hierarchy. But they do not provide additional flows because these new derivations do not commute among themselves. Instead, as expected, they close on a subalgebra of the Virasoro algebra.

Proposition 3.4. The derivations (3.9) have the following commutation relations:

$$\left[\frac{\partial}{\partial \beta_m}, \frac{\partial}{\partial \beta_n}\right] = (m-n)\frac{\partial}{\partial \beta_{m+n}} \tag{3.12}$$

for m, n constrained as in (3.9).

Proof. Again, it will be sufficient to consider the commutator of two derivations acting on a generic Lax operator. The result is

$$\begin{aligned}\left[\frac{\partial}{\partial \beta_m}, \frac{\partial}{\partial \beta_n}\right]\mathcal{L}_j &= +[\mathrm{P}_{<0[s]}([\mathrm{P}_{<0[s]}(S_m), \Theta \widetilde{s}_n \Theta^{-1}]), \mathcal{L}_j) + [\mathrm{P}_{<0[s]}(S_n), [\mathrm{P}_{<0[s]}(S_m), \mathcal{L}_j]] \\ &\quad - [\mathrm{P}_{<0[s]}([\mathrm{P}_{<0[s]}(S_n), \Theta \widetilde{s}_m \Theta^{-1}]), \mathcal{L}_j) - [\mathrm{P}_{<0[s]}(S_m), [\mathrm{P}_{<0[s]}(S_n), \mathcal{L}_j]] \\ &= [\mathrm{P}_{<0[s]}(\Theta[\widetilde{s}_m, \widetilde{s}_n]\Theta^{-1} - [\eth_m^{(s)}, \eth_n^{(s)}]), \mathcal{L}_j] \end{aligned} \tag{3.13}$$

which, using (3.16) and (A.2), proves (3.12). □

Notice that the infinite set of additional symmetries generated by the equations (3.9) and (3.10) have been constructed in a completely representation independent fashion; although it is important to remember that these expressions are only valid in the special gauge chosen in section 2.

For the sake of illustration, we shall write the generators of the first two additional symmetries in terms of the loop-algebra representation of $\mathfrak{g}$, $\mathbb{L}(g) = \mathbb{C}[z, z^{-1}] \otimes g$ (the central term c vanishes). In this representation, $\partial_n = -z^{n+1}\mathrm{d}/\mathrm{d}z$ and $\partial_n^{(s')}$ is given by (A.4). Let us consider a generalized hierarchy of the KdV type, i.e., one for which $s = s_{\mathrm{hom}}$. Then

$$\begin{aligned} \mathrm{P}_{\geqslant 0[s]}(S_{-1}) &= \sum_{j\in E>0} \frac{j}{N_{s'}} t_j \mathrm{P}_{\geqslant 0[s]}(\Theta b_{j-N_{s'}}\Theta^{-1}) \\ \mathrm{P}_{\geqslant 0[s]}(S_0) &= \sum_{j\in E>0} \frac{j}{N_{s'}} t_j \mathrm{P}_{\geqslant 0[s]}(\Theta b_j \Theta^{-1}) - \frac{H_{s'}}{N_{s'}} + \frac{\alpha}{N_{s'}} b_0 \end{aligned} \tag{3.14}$$

where, as before, α is not present if $0 \notin I$. Then, if $k < N_{s'}$, it is straightforward to write the first two generators as

$$-\frac{\partial q(k)}{\partial \beta_{-1}} = \sum_{j\in E>N_{s'}} \frac{j}{N_{s'}} t_j \frac{\partial q(k)}{\partial t_{j-N_{s'}}} - \frac{\mathrm{d}}{\mathrm{d}z}(b_k + q(k)) - \sum_{0<j\in E\leqslant N_{s'}} \frac{j}{N_{s'}} [t_j P_{0[s]}(b_{j-N_{s'}}), \mathcal{L}_k] \tag{3.15}$$

and

$$-\frac{\partial q(k)}{\partial \beta_0} = -\left[z\frac{\mathrm{d}}{\mathrm{d}z} + \frac{H_{s'}}{N_{s'}}, q(k)\right] + \frac{k}{N_{s'}} q(k) + \sum_{j\in E>0} \frac{j}{N_{s'}} t_j \frac{\partial q(k)}{\partial t_j} + \frac{\alpha}{N_{s'}} \frac{\partial q(k)}{\partial t_0}. \tag{3.16}$$

The first one generalizes the infinitesimal generator of the Galilean transformation of the KdV equation (1.5). In fact, in the particular case of the KdV equation, (3.15) is

$$-\frac{\partial u}{\partial \beta_{-1}} = \sum_{j\in \mathbb{Z}>0} \left(j + \frac{1}{2}\right) t_{2j+1} \frac{\partial u}{\partial t_{2j-1}} + 1 \tag{3.17}$$

which, taking $t_j = 0$ for $j > 3$, is just (1.5). Moreover, the result that the transformation generated by β_{-1} is a symmetry only of hierarchies associated with $s = s_{\mathrm{hom}}$ is the generalization of the well known fact that the KdV equation is Galilean invariant, whilst the MKdV equation (which has $s = s'$ being the principal gradation) is not [2].

In order to gain a better understanding of the transformation generated by β_0, let us specify the components of $q(k)$ with respect to an s'-graded basis of $Q(k)$: $q(k) = \sum_{r<k} q^r(k) e_r$, with $[d_{s'}, e_r] = r e_r$. In terms of these components, (3.16) is

$$-\frac{\partial q^r(k)}{\partial \beta_0} = \sum_{j\in E>0} \frac{j}{N'_s} t_j \frac{\partial q^r(k)}{\partial t_j} + \frac{k-r}{N_{s'}} q^r(k) + \frac{\alpha}{N_{s'}} \frac{\partial q^r(k)}{\partial t_0} \tag{3.18}$$

which is the generator of a scaling transformation under which the scaling dimension of $q^r(k)$ is $k - r$ and that of t_j ($j \neq 0$) is j. This scaling symmetry of the generalized hierarchies has been already discussed in [10]. In particular, (3.18) generalizes (1.6).

4. Additional symmetries and tau-functions

When $\mathfrak{g}$ admits vertex-operator representations, *some* of the integrable hierarchies defined by equations (2.3) can be described using the tau-function formalism [11]. In terms of the tau-functions, the hierarchy consists of an infinite set of bilinear equations known as Hirota equations and it is related to one of the integrable hierarchies constructed by Kac and Wakimoto [5]. Consequently, the additional symmetries generated by (3.9) and (3.10) can also be written as transformations of the corresponding tau-functions.

4.1. The tau-function formalism

For the sake of completeness, let us briefly review the construction of integrable hierarchies within the tau-function formalism and their connection with the zero-curvature hierarchies (for more details see [5, 11, 14] or, in general, [4]). The tau-function $\tau_{\mathbf{s}}$, associated with an integrable highest-weight representation $L(\mathbf{s})$ of an affine Kac–Moody algebra $\mathfrak{g}$, is characterized by saying that it lies in the G-orbit of the highest-weight vector $v_{\mathbf{s}}$ with G being the group associated with $\mathfrak{g}$.

Let $\{u_i\}$ and $\{u^i\}$ be dual bases of the larger algebra $\mathfrak{g} \oplus \mathbb{C}d$ with respect to the non-degenerate bi-linear form $(\cdot \mid \cdot)$. It can be shown [5, 27] that $\tau_{\mathbf{s}}$ lies in the G-orbit of $v_{\mathbf{s}}$ if, and only if,

$$\sum u_i \otimes u^i(\tau_{\mathbf{s}} \otimes \tau_{\mathbf{s}}) = (\Lambda_{\mathbf{s}} \mid \Lambda_{\mathbf{s}})\tau_{\mathbf{s}} \otimes \tau_{\mathbf{s}} \tag{4.1}$$

where $\Lambda_{\mathbf{s}}$ is the eigenvalue of $\mathfrak{g}_0(\mathbf{s})$ acting on $v_{\mathbf{s}}$. Furthermore, the condition (4.1) is also equivalent to the statement that $\tau_{\mathbf{s}} \otimes \tau_{\mathbf{s}} \in L(2\mathbf{s})$. It follows from the definition of the action of a group on a tensor product that, for the representation $L(\mathbf{s})$

$$\tau_{\mathbf{s}} = \bigotimes_{i=0}^{r} \{\tau_i^{\otimes s_i}\} \tag{4.2}$$

where τ_i is the tau-function corresponding to the fundamental representation with $s_j = \delta_{j,i}$.

When the representation $L(\mathbf{s})$ is a vertex-operator representation, (4.1) can be interpreted as a set of differential equations on the tau-functions. In fact, they are precisely the Hirota equations of an integrable hierarchy. Let us restrict ourselves to cases where $\mathfrak{g}$ is the untwisted affinization of a simply-laced algebra (i.e., g is of A, D or E type). In that case, level-one representations (or *basic* representations, those for which $s_j = \delta_{j,i}$ for some i with unit Kac label) are isomorphic to the Fock space of any of the Heisenberg subalgebras of $\mathfrak{g}$ which are classified by the conjugacy classes of the Weyl group of g.

The Heisenberg subalgebra $\mathfrak{s}_w$, associated with some element of the Weyl group (say, w up to conjugacy) is realized on the Fock space $\mathbb{C}[x_j;\, j \in E > 0]$ in the standard way

$$c = 1 \qquad \text{and} \qquad b_j = \begin{cases} \partial/\partial x_j & \text{for } j > 0 \\ |j| x_{|j|} & \text{for } j < 0. \end{cases} \tag{4.3}$$

A different treatment is required for the zero-graded elements of $\mathfrak{s}_w$, which correspond to the invariant subspace of w. These zero-modes are represented on the space

$$\mathbb{C}(Q) = \{\exp(\beta \cdot x_0);\ \beta \in Q\} \tag{4.4}$$

where Q is the root lattice of g projected onto the invariant subspace of w; b_0 acts as $\partial/\partial x_0$.

The level-one representation is isomorphic to $\mathbb{C}[x_j] \otimes \mathcal{V}$ where $\mathcal{V} = \mathbb{C}(Q) \otimes V$ is the zero-mode space. Here, V is an additional finite-dimensional vector space [8, 28] which is trivial ($\dim(V) = 1$) for the cases relevant to our discussion [11]. The elements of $\mathfrak{g}$ not in $\mathfrak{s}_w$ are the modes of vertex operators and the derivation $d_{\mathbf{s}_w}$ is related to the zero-mode of the Sugawara current.

Summarizing, the vertex-operator representation of $L(\mathbf{s})$ is realized on the tensor product of fundamental representations where s_i gives the multiplicity of the ith fundamental representation in the product (so any non-zero s_i corresponds to $k_i = k_i^\vee = 1$). They will be carried by a tensor product of the Fock spaces

$$\bigotimes_{i=1}^{N_s}\{\mathbb{C}[x_j^{(i)}, j \in E > 0] \otimes \mathcal{V}\} \tag{4.5}$$

where $x_j^{(i)}$ indicates the Fock space variables of the ith space in the tensor product and $x_j \equiv \sum_{i=1}^{N_s} x_j^{(i)}$.

In the tau-function formalism of [5], a hierarchy of Hirota equations is associated with $\{g, w, s\}$, i.e. a simply-laced finite Lie algebra g, an element w of the Weyl group of g (up to conjugacy) and a vertex-operator realization of $L(s)$ where $s_i = 0$ if $k_i \neq 1$. Finally, the connection between the zero-curvature and the tau-function formalism is established by the following theorem.

Theorem 4.1. (Theorem 5.1 of [11].) There exists a map, from solutions of the Kac–Wakimoto hierarchy, associated with the data $\{g, w, s\}$ (with the gradation associated with the Heisenberg subalgebra s_w satisfying $s \preceq s_w$ and also $s_i \neq 0$ only if $k_i = 1$) and a zero-curvature hierarchy associated with $s' = s_w$ given by

$$\Theta^{-1} \cdot v_s = \tau_s(x + t)/\tau_s^{(0)}(t) \tag{4.6}$$

where $\Theta \in U_-(s)$ gives $q(k)$ via (2.8) and $\tau_s^{(0)}(x)$ is the x_0-independent component of τ, i.e., the component corresponding to $\beta = 0$ in (4.4).

Notice that not all the zero-curvature hierarchies can be related to tau-functions (g has to be simply-laced and s must correspond to products of level-one representations). Conversely, not all the Kac–Wakimoto hierarchies can be related to zero-curvature hierarchies because of the condition $s \preceq s_w$.

4.2. Additional symmetries of the tau-functions

In a vertex-operator representation, the generators of the Virasoro algebra can be realized in terms of the elements of the Heisenberg subalgebra through the Sugawara construction (see, for example, [29])

$$\begin{aligned} \mathfrak{d}_n^{(s')} \mapsto L_n^{(s')} &= \frac{1}{2N_{s'}} \sum_{i+j=nN_{s'}} : b_i b_j : + \eta_{s'} \delta_{n,0} \\ \eta_{s'} &= \frac{1}{4N_{s'}^2} \sum_{j \in I} j(N_{s'} - j) \end{aligned} \tag{4.7}$$

where : : indicates that the product of elements of s is 'normal-ordered'. Consequently, acting on the Fock space, the generators $L_n^{(s')}$ are second-order differential operators; it will be convenient to write them as $L_n^{(s')} \equiv L_n^{(s')}(\{x_i\}, \{\partial_j\})$ where $\partial_j \equiv \partial/\partial x_j$.

Taking into account (3.10) and (4.6), one can easily derive the action of the derivations (3.9) on the tau-functions

$$\frac{\partial \Theta^{-1}}{\partial \beta_n} \cdot v_s = -\Theta^{-1} \mathrm{P}_{<0[s]}(S_n) \cdot v_s = \frac{1}{\tau_s^{(0)}(t)} \frac{\partial \tau_s(x+t)}{\partial \beta_n} - \frac{\tau_s(x+t)}{[\tau_s^{(0)}(t)]^2} \frac{\partial \tau_s^{(0)}(t)}{\partial \beta_n}. \tag{4.8}$$

Now, using (4.7), it is straightforward to see that

$$\widetilde{S}_n \mapsto L_n^{(s')}(\{x_i + t_i\}, \{\partial_j + \alpha\delta_{j,0}\}) + \left(\lambda - \frac{\alpha^2}{2N_{s'}}\right)\delta_{n,0}. \tag{4.9}$$

Therefore,

$$\begin{aligned} S_n \cdot v_s &= P_{\leqslant 0}(S_n) \cdot v_s = (\Theta \widetilde{S}_n \Theta^{-1} - \partial_n^{(s)}) \cdot v_s \\ &= \frac{1}{\tau_s^{(0)}(t)} \Theta \left[L_n^{(s')}(\{x_i + t_i\}, \{\partial_j + \alpha\delta_{j,0}\}) + \left(\lambda - \frac{\alpha^2}{2N_{s'}}\right)\delta_{n,0} \right] \tau_s(x+t) - \eta_s \delta_{n,0} \end{aligned} \tag{4.10}$$

the x-independent part corresponding to $P_{0[s]}(S_n) \cdot v_s$. Now, the comparison of (4.8) and (4.10) allows one to prove the following proposition.

Proposition 4.2. In the tau-function formalism, the additional symmetries of the hierarchy are generated by the derivations

$$\frac{\partial \tau_s(x)}{\partial \beta_n} = -(L_n^{(s')}(\{x_i\}, \{\partial_j + \alpha\delta_{j,0}\}) + \mu\delta_{n,0})\tau_s(x) \tag{4.11}$$

where

$$n \in \mathbb{Z} \geqslant \begin{cases} -1 & \text{if } s = s_{\text{hom}} \\ 0 & \text{if } s \succ s_{\text{hom}} \end{cases}$$

and α is an arbitrary constant that is not present if $0 \notin E$. In general, μ is also an arbitrary constant but it has to vanish if $s = s_{\text{hom}}$ as required by the commutation relation $[\partial/\partial\beta_1, \partial/\partial\beta_{-1}] = 2\partial/\partial\beta_0$.

As we have discussed above, the vertex-operator representation of $L(s)$ is realized on the tensor product of fundamental representations and the tau-function is also a tensor product of 'fundamental' tau-functions (4.2). Consequently, the derivations (4.11) act on these 'fundamental' tau-functions as

$$\frac{\partial \tau_i}{\partial \beta_n} = -(L_n^{(s')}(\{x_j\}, \{\partial_j + \alpha_i\delta_{j,0}\}) + \mu_i\delta_{n,0})\tau_i(x) \tag{4.12}$$

with

$$n \in \mathbb{Z} \geqslant \begin{cases} -1 & \text{if } s_j = \delta_{j,0} \\ 0 & \text{otherwise.} \end{cases} \tag{4.13}$$

Of course, there is one equation for each component $s_i \neq 0$ (having $k_i = 1$). Again, α_i and μ_i are arbitrary constants in general, but all the μ_is have to vanish if $s = s_{\text{hom}}$. These results agree with those obtained in [14], where it has been proven that if τ_s is a solution of the Hirota equations (4.1), so is $\tau_s + \epsilon L_n \tau_s$, $\epsilon \ll 1$.

Finally, let us check again, in this formalism, that the transformation generated by (4.12) with $n = -1$ actually generalizes the Galilean transformation of the KdV equation. For the KdV equation, w is the Coxeter element, $\mathfrak{g} = A_1^{(1)}$, $s_w = (1, 1)$, $s = (1, 0)$ and there is only one (scalar) tau-function τ. Then,

$$-\frac{\partial \tau}{\partial \beta_{-1}} = \left(\sum_{j \in \mathbb{Z}>0} (j + \tfrac{1}{2}) t_{2j+1} \frac{\partial}{\partial t_{2j-1}} + \frac{t_1^2}{4} \right) \tau. \tag{4.14}$$

Therefore, using equation (1.2), one recovers the action of this derivation on the original variable of the KdV equation (3.17).

5. Generalizing the string equation

As explained in the introduction, one of the main recent motivations to study integrable hierarchies of partial-differential equations is their importance in the matrix-model formulation of two-dimensional quantum and topological gravity. For the multi-matrix model, after applying the double scaling limit [30], the string equation has the general form [31]

$$\left[\frac{\partial}{\partial z} - P, \mathcal{L}\right] = 0 \tag{5.1}$$

where $\mathcal{L}$ is the Lax operator (in matrix form) of the A_n-KdV hierarchy.

Given that the zero-curvature hierarchies of [9–11] generalize the Drinfel'd–Sokolov hierarchies and share the same structure, it is very tempting to consider the possibility that some of them could also describe interesting physical systems coupled to quantum or topological gravity in a similar way. For that to be the case, then, at the very least, the hierarchy must admit a generalization of the string equation (5.1).

The possibility of imposing additional constraints of the form (5.1) is ensured by the existence of the additional symmetries [12]. Recall that, in the loop-algebra representation, $\eth_n = -z^{n+1}\mathrm{d}/\mathrm{d}z$; hence, the invariance of the Lax operator under the infinitesimal generator (3.9) with $n = -1$

$$-\frac{\partial \mathcal{L}_k}{\partial \beta_{-1}} = \left[\frac{\mathrm{d}}{\mathrm{d}z} + \mathrm{P}_{\geqslant 0[s]}(S_{-1}), \mathcal{L}_k\right] = 0 \tag{5.2}$$

for some $\mathcal{L}_k \equiv L$, is precisely a generalization of (5.1). Moreover, one can check that the condition

$$\mathrm{P}_{<0[s]}(S_{-1}) = 0 \tag{5.3}$$

is compatible with the hierarchy in the sense that it is preserved by all the flows. Obviously, it induces the constraint (5.2) and it is the natural choice for the 'generalized string equation' for any zero-curvature hierarchy of the KdV type†. Moreover, when the hierarchy can be written in terms of tau-functions, the constraint (5.3) translates into an L_{-1} constraint for the (unique) tau-function

$$L_{-1}^{(s')}(\{x_i\}, \{\partial_j + \alpha\delta_{j,0}\})\tau = 0 \tag{5.4}$$

according to (4.11).

It is well known that the string equation, together with the recurrence relations of the relevant hierarchy, induces an infinite set of Virasoro constraints [20, 21]. In the generalized case we are discussing, the generalized string equation (5.3) also induces an infinite set of constraints. To prove this, we restrict ourselves again to the loop-algebra representation. The crucial observation is that $S_n = z^j S_{n-j}$ for any $n, j \in \mathbb{Z}$. Therefore, the generalized string equation (5.3) implies the following infinite tower of constraints

$$\mathrm{P}_{<0[s]}(S_n) = 0 \qquad n \in \mathbb{Z} \geqslant -1. \tag{5.5}$$

† If the hierarchy is not of the KdV type, i.e., $s \neq s_{\mathrm{hom}}$ then the natural generalization of the string equation would be $\mathrm{P}_{<0[s]}(S_0) = 0$ in agreement with the results of the unitary matrix models.

Even though we have used the loop-algebra representation to prove (5.5), the representation independence of the zero-curvature hierarchies and the fact that equations (5.3) and (5.5) are explicitly independent of the centre, ensure that the result is completely general. On the tau-function, (5.5) are just the Virasoro constraints as expected

$$L_n^{(s')}(\{x_i\}, \{\partial_j + \alpha\delta_{j,0}\})\tau = 0 \qquad n \geqslant -1. \tag{5.6}$$

For the original case of the KdV hierarchy, which describes ordinary quantum gravity, the set of Virasoro constraints (5.6) are complete in the sense that they are equivalent to imposing the string equation (5.4) along with the fact that τ is the tau-function of the hierarchy. For the more general hierarchies this is not the case and the generalized string equation plus the hierarchy is equivalent to the Virasoro constraints (5.6), as we have shown, plus some additional constraints. It is thought that these additional constraints satisfy a subalgebra of a $\mathcal{W}$-algebra. The main evidence for this belief comes from the study of the KP hierarchy which contains the Drinfel'd–Sokolov A_n-hierarchies as reductions. In this case, and using the Grassmannian approach, it has been proven that the string equation of the KP hierarchy induces an infinite set of constraints satisfying a subalgebra of the $\mathcal{W}_{1+\infty}$-algebra. Moreover, the algebra satisfied by the constraints reduces to a subalgebra of the classical W_n-algebra when the KP hierarchy is reduced to the Drinfel'd–Sokolov A_{n-1}-hierarchy [21]. Directly in terms of the Lax operator approach, and again considering reductions of the KP hierarchy, it has also been proven that the string equation induces an infinite set of constraints spanning a subalgebra of the classical W_3-algebra in the case of the Drinfel'd–Sokolov A_2-hierarchy [32].

The generalized string equation (5.3) does imply that the quantities $\mathrm{P}_{<0[s]}(S_{m_1}\cdots S_{m_n}) = 0$ for $\sum_{i=1}^{n} m_i \geqslant -n$; however, we have not managed to write these equations as constraints directly on the tau-function and show that it satisfies a $\mathcal{W}$-algebra. It is clear, though, that these additional constraints are not related to additional symmetries of the hierarchy†. In the absense of a direct construction of the constraints on the tau-function we shall limit ourselves to some observations.

First of all, let us point out that whatever the additional constraints are, they have to be consistent with the Virasoro constraints (5.6) and so form a closed algebra with this subalgebra of the Virasoro algebra. So, the most natural guess is that they satisfy a subalgebra of the $\mathcal{W}$-algebra associated with the Casimirs of the relevant finite Lie algebra g, being realized in terms of the Heisenberg subalgebra s through a generalized Sugawara construction; the generators being differential operators $W_n^{(s')}(\{x_i\}, \{\partial_j\})$. We shall now prove that these additional constraints would be compatible with the hierarchy. Let us consider

$$R_n \cdot v_s = \Theta W_n^{(s')}\left(\{\partial_i\}, \{x_j + t_j\}\right)\Theta^{-1} \cdot v_s = 0 \tag{5.7}$$

which follows from a $\mathcal{W}$-constraint on the tau-function

$$W_n^{(s')}\left(\{\partial_i\}, \{x_j\}\right) \cdot \tau_s = 0. \tag{5.8}$$

The time evolution of (5.7) is

$$\begin{aligned}\frac{\partial R_n}{\partial t_j}\cdot v_s &= \left(\Theta\frac{\partial}{\partial t_j} W_n^{(s')}(\{\partial_i\}, \{x_j + t_j\})\Theta^{-1} - [\mathrm{P}_{<0[s]}(\Theta b_j \Theta^{-1}), \Theta W_n^{(s')}(\{\partial_i\}, \{x_j\})\Theta^{-1}]\right)\cdot v_s \\ &= [\mathrm{P}_{\geqslant 0[s]}(\Theta b_j \Theta^{-1}), R_n] \cdot v_s = 0 \end{aligned} \tag{5.9}$$

† See [14] for similar comments about additional constraints and additional symmetries within the tau-function approach.

for any n, where we have used that $b_j \equiv \partial/\partial x_j$ for $j \in E \geqslant 0$ (5.7) and the fact that v_s is annihilated by $\mathfrak{g}_{>0[s]}$ and an eigenvector of $\mathfrak{g}_{0[s]}$. Obviously, the reason for the consistency of (5.7) with the hierarchy is just the identity

$$\left[\frac{\partial}{\partial t_j} - b_j, W_n^{(s')}(\{\partial_i\}, \{x_j + t_j\})\right] = 0 \tag{5.11}$$

and, of course, we could add arbitrary constant elements of $\mathfrak{s}$.

In the absence of a proof, we conclude this section by making the natural conjecture that, for the cases where a tau-function formalism exists, the generalized string equation (5.3) induces an infinite set of constraints on the tau-function which satisfy part of the $\mathcal{W}$-algebra corresponding to the Casimirs of g for which there is a tower of generators for each exponent of g. This conjecture can be taken as a starting point to investigate the possibility that some generalized integrable hierarchies could describe two-dimensional physical systems including quantum gravity [33].

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Appendix

In this appendix we review the semi-direct product of the Virasoro algebra with $\mathfrak{g}$ for arbitrary g rotations. Our rotation and approach follows [34]. Let us choose a basis $\{E_{n\delta}^{(i)}; i = 1, \ldots, r\}$ of the root space $\mathfrak{g}'_{n\delta}$ for $n \in \mathbb{Z} \neq 0$ and $E_{\alpha+m\delta}$ of $\mathfrak{g}'_{\alpha+m\delta}$ for $\alpha \in \overline{\Delta}$ and $m \in \mathbb{Z}$ and define the following set of derivations labelled by a gradation $\mathbf{s}$:

$$\begin{aligned} &[\mathfrak{d}_m^{(s)}, E_{n\delta}^{(i)}] = -nE_{(m+n)\delta}^{(i)} \qquad [\mathfrak{d}_m^{(s)}, \mathfrak{h}] = 0 \\ &[\mathfrak{d}_m^{(s)}, E_{\pm\alpha_i+n\delta}] = -\left(n \pm \frac{s_i}{N_s}\right) E_{\pm\alpha_i+(m+n)\delta} \qquad i = 1, \ldots, r \end{aligned} \tag{A.1}$$

and

$$[\mathfrak{d}_m^{(s)}, \mathfrak{d}_n^{(s)}] = (m-n)\mathfrak{d}_{m+n}^{(s)} + \frac{\widetilde{c}_v}{12} m(m^2-1)\delta_{m+n,0}. \tag{A.2}$$

These derivations span a Virasoro algebra

$$\mathfrak{Vir} = \bigoplus_{m\in\mathbb{Z}} \mathbb{C}\mathfrak{d}_m^{(s)} \tag{A.3}$$

and (A.1), together with $[\widetilde{c}_v, \mathfrak{g}] = 0$, define the *semi-direct product* of $\mathfrak{Vir}$ and $\mathfrak{g}$, sometimes denoted as $\mathfrak{Vir} \rhd\!\!< g$.

Let $\{\mathfrak{d}_n; n \in \mathbb{Z}\}$ be the Virasoro generators labelled by the homogeneous gradation. Then, it is easy to prove that [34]

$$\mathfrak{d}_n^{(s)} = \mathfrak{d}_n - \frac{H_s^{(n)}}{N_s} + c\frac{(H_s \mid H_s)}{2N_s^2}\delta_{n,0} \tag{A.4}$$

where $H_s^{(n)}$ is an element of $\mathfrak{g}_{n\delta}$ such that $[H_s^{(n)}, E_{\pm\alpha_i+m\delta}] = \pm s_i E_{\pm\alpha_i+(m+n)\delta}$; with $H_s^{(0)} = H_s$ an element of the Cartan subalgebra. Furthermore, $\mathfrak{d}_n^{(s)} - \mathfrak{d}_n^{(s')} \in \mathfrak{g}$. It follows from (A.4) that $\mathfrak{d}_0 = -d$ and

$$\mathfrak{d}_0^{(s)} = -\frac{d_s}{N_s} + c\frac{(H_s \mid H_s)}{2N_s^2}. \tag{A.5}$$

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